JITTER CHARACTERIZATION IN SPACE MECHANISMS

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ABSTRACT

Coupling between degrees of freedom in linear elastic structural systems is studied to mitigate jitter and improve pointing and tracking. Critical points at which the response to external loads decouple are classified and systematically located in flexure mounted systems. Conditions for the existence of such points are established. Following the introduction of degree-of-freedom coupling measures, unique critical stationary points where coupling is minimum are found. When mechanisms are designed such that resultant inertial loads and/or actuation loads are channeled through these characteristic points, higher performance is achieved, effectively minimizing jitter induced by imbalances from imperfections due to manufacturing or assembly. The balancing of an actuated mirror is used as an example to illustrate jitter minimization and pointing improvement.

Key words: coupling, jitter, balancing, center of rotation.

1. INTRODUCTION

The quality of data recovered from precision-instruments depends primarily on the basic structural properties of the hardware being used. Instruments deform in response to thermal loads, moisture dissipation, jitter disturbance etc., introducing errors and unwanted noise in measurements. For example, electro-optics systems are susceptible to thermal excursions, mounting and inertial loads making automated image processing less than perfect. In some applications, delicate mirrors may have to be steered in fast motion to overcome jitter and other disturbances using control schemes, over some bandwidth which depends on how the degrees of freedom couple. In other applications, jitter isolation of mechanisms is essential. For example, loads due to jitter in a mechanism of a subsystem can produce negative effects on the performance of other neighboring subsystems. Therefore, attention to causes and effects of ”cross-talk” between degrees of freedom is critical for instruments and equipments intended for superior performance. One common denominator in precision design of optical systems, cryocoolers, scanners, and other space equipment is the mounting of a stiff system on flexures. These flexures, found in many shapes and layouts, are precisely tuned to deliver a function, such as the reduction of imparted thermal loads, isolation of jittery mechanical subsystems or aligning precise single-degree of freedom actuation. Flexure design addresses the response of particular degrees of freedom under certain load conditions. Coupling between separate DOFs is often unavoidable, and the more the system is complicated, the more the mitigation of certain behaviors becomes difficult.

Unfortunately, the subject of coupling between separate degrees of freedom lacks development in structural analysis and design. The first time where coupling between modes of deformation comes to light is in elementary mechanics of solids texts (e.g. [1]) where the issue of shear center is introduced. Historically, the topic is taught to aid steel frame design. Because the finite element method (FEM) has become widely used, the topic is now obscure and many practicing engineers would not connect the textbook subject with design practices. Also, prior to the use of the finite element method, use of flexural pivots in mechanisms has led to developments that revolved around specific applications, [3], [4], and [5], and the topic was not addressed from a fundamental perspective. This is why there is an attempt to formulate some fundamentals herein. It will be shown that general coupling between degrees of freedom is deviatoric in nature, of which the concept of shear center is a particular case.

2. BASIC DEFINITIONS

Consider a linear elastic structure which has been reduced to six degrees of freedom (DOF) and subjected to a resultant generalized force (i.e. force or moment). Such reduction (see e.g. [2]), referred to as static condensation in FEM, introduces no approximations and is therefore exact from a stiffness perspective. Despite the loss of higher modal content, coupling between degrees of freedom is preserved from the original structure because of the unicity of condensation. For such system, there are four basic levels of coupling that can be encountered: The first case concerns structures for which there exists a point P in 3-D space through-which an arbitrary 1-DOF generalized force generates pure motion in the force direction. The second case concerns similar structures for which this time there exists a point P in 3-D space through-which an arbitrary 1-DOF generalized force generates motion of similar kind, that is forces produces only translations and moments generate rotations only. In essence, the motions are of the same type as the load but not necessarily co-linear. The third case is of a structure for which there exists a point C_s in a plane A at which at least a pair of translation-rotation associated with A decouple . Finally, we can think of the case where no point exists where translations decouple from rotations. Having classified these possibilities, we may state several definitions:

Definition 1 *A point in 3D space is called* pure center of rotation *of a particular structure, if loads and motions at this point are uncoupled in the six degrees of freedom.*

The second possibility presents the less restrictive case of rotation-translation decoupling instead of a DOF by DOF decoupling. For such possibility we introduce the following definition:

Definition 2 *The* **center of rotation** C_r *of a structure is a point in 3D space through which, when loads are applied, the translation and rotation modes of deformation decouple.*

Finally, when uncoupled response is only possible for degrees of freedom associated with a particular plane, as identified in the third possibility, we have the classical *shear center* of structural mechanics. Hence we may introduce the following definition:

Definition 3 *The* center of shear *or* shear center C_s *of a structure is a point of a plane* A *where at least a pair of translation-rotation degrees of freedom associated with* A *decouple.*

Flexural systems with pure centers of rotation are important to designers and have been the basis of patent protection. The center of rotation, subject of Definition 2, is a generalization of the concept of *pure center of rotation*. Finally, we will define the point of minimal translation-rotation coupling for general structures that do not necessarily possess a center of rotation. Such structures may be coined "hypo-symmetric" as coupling goes hand-in hand with symmetry.

3. TRANSFORMATIONS BETWEEN RIGIDLY CONNECTED POINTS

In this section, we focus attention on the flexibility transformation between rigidly connected points. Consider the structure shown in Figure 1. Two components are shown in the Figure, a relatively rigid body denoted by B_r and a set of two flexures, which assembly is denoted by B_f . B_f is less stiff than B_r . A practical way to differentiate between B_f and B_r would be simply to have the natural frequencies of B_f and B_r differ by more than one order of magnitude. Also shown in Figure 1 is the center of mass of the structure, C_G , the center of rotation C_R which position is for now unknown, and a point C_i through which we intend to apply some arbitrary loads. The point C_i can be chosen arbitrarily. Because of the

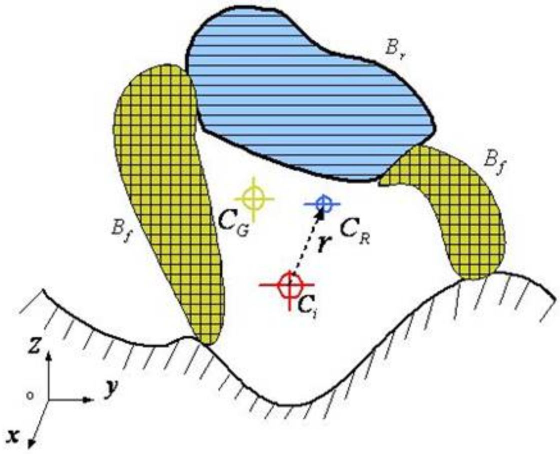

Figure 1. Typical structure with two components: stiff body B_r and a relatively flexible subsystem B_f

relative stiffness between B_r and B_f , we can replace B_r by a rigid body. This rigid body has an added link which attach to C_i from any point in the rigid body. Figure 2 illustrates such transformation. Denoting by u_i and u_r the displacement vectors of points C_i and C_r respectively, we can write

$$u_r = u_i + d_i, \tag{1}$$

where ${u_r}^T = (u_{xr}, u_{yr}, u_{zr}, \theta_{xr}, \theta_{yr}, \theta_{zr})$, ${u_i}^T = (u_{xi}, u_{yi}, u_{zi}, \theta_{xi}, \theta_{yi}, \theta_{zi})$ and d_i is given by

$$d_i = \begin{pmatrix} \omega_i r \\ 0 \end{pmatrix}. \tag{2}$$

In Equation (2), the vector r is given by $r^T = (r_x, r_y, r_z)$ and ω_i is the skew symmetric 3×3 rotation matrix given by

$$\begin{pmatrix} 0 & -\theta_{zi} & \theta_{yi} \\ \theta_{zi} & 0 & -\theta_{xi} \\ -\theta_{yi} & \theta_{xi} & 0 \end{pmatrix}. \tag{3}$$

the displacement at C_r can be expressed in terms of the displacement at point C_i in the following matrix form

$$u_r = (I_6 + M)u_i, \tag{4}$$

where I_6 is the six by six identity matrix and M is the two-by-two block matrix given by

$$\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}. \tag{5}$$

The non-zero three-by-three matrix in Equation (5) locates C_r from point C_i and is given by

$$T = \begin{pmatrix} 0 & r_z & -r_y \\ -r_z & 0 & r_x \\ r_y & -r_x & 0 \end{pmatrix}. \tag{6}$$

Consider now an arbitrary generalized load P_i, applied at C_i, one can remove the link connecting C_i to the rigid body of Figure 2 and replace it with a new link connecting

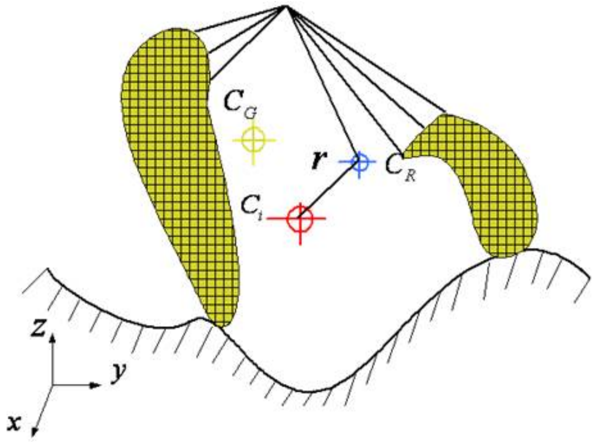

Figure 2. Structure of Figure 1 with B_r replaced by a rigid body with a link extending to C_i

C_r this time with a load P_r . Using a free body diagram of the rigid body, the loads at C_r and C_i are related through

$$P_r^T = P_i^T + (0, -r \otimes P_{i_1})^T , \tag{7}$$

where P_{i_1} is the translational component of P_i^T . Here the load vector at point C_i is decomposed into translational and rotation components, i.e. $P_i^T = [P_{i_1}, P_{i_2}]^T$. Equation (7) can be further reduced in a form similar to that of Equation (4):

$$P_r = (I_6 - M^T)P_i. \tag{8}$$

Finally, let us consider the problem of Figure 2. One can reduce the problem to six degrees of freedom where the load and displacements at C_i are related through:

$$F_i P_i = u_i. \tag{9}$$

Substituting Equations (4) and (8) into Equation (9), we obtain

$$F_i (I_6 - M^T)^{-1} P_r = (I_6 + M)^{-1} u_r. \tag{10}$$

It can be easily shown that $(I_6 - M^T)^{-1} = (I_6 + M)^T$. Substituting this identity into Equation (10) after pre-multiplying both sides by $(I_6 + M)$ and making use of the symmetry of the flexibility matrix, we obtain:

$$(I_6 + M)F_i(I_6 + M^T)P_r = u_r. \tag{11}$$

Because of the unicity of the flexibility matrix, we finally have the flexibility matrix at the point C_r given by

$$F_r = (I_6 + M)F_i(I_6 + M^T), \tag{12}$$

where

$$F_r P_r = u_r. \tag{13}$$

Equation 12 is the load transformation between rigidly connected points in an arbitrary structure, and Equation 13 is the equilibrium statement at an arbitrary point C_r. Furthermore, the stiffness matrix transforms between C_i and C_r via

$$K_r = (I_6 - M^T)K_i(I_6 - M) \tag{14}$$

where $K_i = F_i^{-1}$ and $K_r = F_r^{-1}$. Let us define the *transvection matrix* $S = I_6 - M$, also called shear matrix in mathematics and referred to as the rigid body matrix in mechanics. We introduce the following key transformations in terms of S:

Lemma 1 *Transformations between two rigidly connected points C_i and C_r are given by:*

$$\textit{Stiffness: } K_i \longrightarrow K_r = \psi(Ki) = S^T K_i S, \tag{15}$$

$$\textit{Flexibility: } F_i \longrightarrow F_r = \phi(Fi) = S^{-1} F_i S^{-T}, \tag{16}$$

$$\textit{Displacements: } u_i \longrightarrow u_r = \tau(u_i) = S^{-1} u_i, \tag{17}$$

$$\textit{Loads: } P_i \longrightarrow P_r = \eta(u_i) = S^T P_i, \tag{18}$$

where S is the transvection matrix from C_i to C_r.

Before moving forward, we summarize some key properties of transvections. Let us denote by ${}^{ij}S$ the transvection from point C_i to point C_j. The first property that one can immediately verify is:

$${}^{ii}S = I_6, \tag{19}$$

which states that when C_i and C_j are coincident, the transvection is the identity matrix. It is also shown that

$${}^{ij}S\,{}^{ji}S = {}^{ji}S\,{}^{ij}S = I_6, \tag{20}$$

which states that ${}^{ij}S^{-1} = {}^{ji}S$. That is switching the order of the indices produces the inverse transvection. Another important result relies on the fact that transvections commute. Given two transvection ${}^{ij}S$ and ${}^{rq}S$ between two pairs of points (C_i, C_j), and (C_r, C_q), we have:

$${}^{ij}S\,{}^{rq}S = {}^{rq}S\,{}^{ij}S, \tag{21}$$

Next, a transvection and its inverse have the identity matrix as a median. We have

$$\frac{1}{2}\left({}^{ij}S + {}^{ji}S\right) = I_6. \tag{22}$$

Also, given three rigidly connected points, C_i, C_r, C_q, and denoting by ${}^{ir}S$, ${}^{rq}S$ and, ${}^{iq}S$ the transvections between (C_i, C_r), (C_r, C_q), and (C_i, C_q) respectively, we have:

$${}^{ir}S\,{}^{rq}S = {}^{iq}S = {}^{ir}S + {}^{rq}S - I_6. \tag{23}$$

The first equality states the transitivity property of the transvection operator, while the second equality is the linearization statement which expresses that the transvectional operator remains linear under composite transformations. Finally it can verified that transcvections commute with symmetric matrices. For example, we have

$$K\,{}^{iq}S = {}^{qi}S\,K \tag{24}$$

And finally, in order to address coupling, we need to partition the flexibility matrices at points C_i and C_r into translational and rotational block matrices:

$$F_i = \begin{pmatrix} f_{i_{11}} & f_{i_{12}} \\ f_{i_{12}}^T & f_{i_{22}} \end{pmatrix} \text{ and } F_r = \begin{pmatrix} f_{r_{11}} & f_{r_{12}} \\ f_{r_{12}}^T & f_{r_{22}} \end{pmatrix}. \quad (25)$$

Using equations 12 and 5, we have

$$\begin{aligned} f_{r_{11}} &= f_{i_{11}} + \left(T f_{i_{12}}^T + f_{i_{12}} T^T\right) + T^T f_{i_{22}} T \\ f_{r_{12}} &= f_{i_{12}} + T f_{i_{22}} \\ f_{r_{22}} &= f_{i_{22}}. \end{aligned} \quad (26)$$

The last equality in Equation 26 shows that the rotation-rotation flexibility matrix is unchanged. Also, taking the trace of the second Equation, and noticing that $tr(f_{i22})$ is nill, leads to the invariance results of the following lemma:

Lemma 2 *The trace of the translation-rotation coupling matrix f_{12} and the rotation-rotation flexibility matrix f_{22} are invariant through a flexibility transformation ϕ.*

Similarly, if we partition the stiffness matrix into translational and rotational blocks:

$$K_i = \begin{pmatrix} k_{i_{11}} & k_{i_{12}} \\ k_{i_{12}}^T & k_{i_{22}} \end{pmatrix} \text{ and } K_r = \begin{pmatrix} k_{r_{11}} & k_{r_{12}} \\ k_{r_{12}}^T & k_{r_{22}} \end{pmatrix}, \quad (27)$$

and use equations 14 and 5, to derive similar relations to the flexibility ones expressed in Equations 26:

$$\begin{aligned} k_{r_{11}} &= k_{i_{11}} \\ k_{r_{12}} &= k_{i_{12}} - k_{i_{11}} T \\ k_{r_{22}} &= k_{i_{22}} + \left(T k_{i_{12}} - k_{i_{12}}^T T\right) + T^T k_{i_{11}} T. \end{aligned} \quad (28)$$

For the invariance result, we notice that the roles of f_{11} and f_{22} are switched, and therefore we have the result:

Lemma 3 *The trace of the translation-rotation coupling stiffness matrix k_{12} and the translation-translation stiffness matrix k_{11} are invariant through a stiffness transformation ψ.*

4. CENTER OF ROTATION

We introduce the decoupling requirement specified in the definition and require that F_r be block diagonal. Equations (26) are reduced to

$$\begin{aligned} f_{r_{11}} &= f_{i_{11}} + \left(T f_{i_{12}}^T + f_{i_{12}} T^T\right) + T^T f_{i_{22}} T \\ f_{i_{12}} + T f_{i_{22}} &= f_{i_{12}}^T - f_{i_{22}} T = 0 \\ f_{r_{22}} &= f_{i_{22}}. \end{aligned} \quad (29)$$

The first equation gives the translation compliance transformation. The first equality of the second equation holds true if and only if

$$T = f_{i_{22}}^{-1} f_{i_{12}}^T = -f_{i_{12}} f_{i_{22}}^{-1}, \quad (30)$$

and finally, the last equation gives the rotational DOF compliance matrix at C_r. Note that F_r is block diagonal, which means $f_{r_{11}}$ and $f_{r_{22}}$ may have off-diagonal terms but no coupling between translations and rotation exist anymore. Next a necessary and sufficient condition for a center of rotation to exist may be established:

Theorem 1 *A center of rotation exists for a particular structure if and only if $f_{i_{12}} = f_{i_{22}}^{-1}\left(-f_{i_{12}}^T\right) f_{i_{22}}$.*

Proof. First observe that for any square matrix M, $M + M^T = 0 \iff M$ is antisymmetric and $M_{ii} = 0$ for $i = 1..3$. Because T falls in this category, using the right hand side of the first equality in Equation 30, we have $\left(f_{i_{22}}^{-1} f_{i_{12}}^T\right)^T + \left(f_{i_{22}}^{-1} f_{i_{12}}^T\right) = 0$. After several manipulations, and making use of the symmetry of $f_{i_{22}}$ the similarity statement of the theorem is shown. □

In practice, direct evaluation of the matrix T may or may not lead to a solution. However the statement in the theorem provides the necessary test that will establish the existence or non-existence of a center of rotation. An important necessary condition for the existence of a center of rotation may be stated as follows:

Theorem 2 *For a center of rotation to exist, the following statements must hold true:*

1. *The first and third invariants of the coupling matrix $f_{i_{12}}$ at any point C_i must vanish.*
2. *At any point C_i, $f_{i_{12}}^3 - \frac{1}{2} tr(f_{i_{12}}^2) f_{i_{12}} = 0$.*

Proof. To prove the first statement, from Equation 30 we have $f_{i_{12}} = -T f_{i_{22}} T$, substitute T from Equation 6 and account for the fact that $f_{i_{22}}$ has a symmetric form, it can be shown that the eigenvalues of $f_{i_{12}}$ are $\left\{0, \pm\sqrt{\frac{1}{2}\ tr\left(f_{i_{12}}^2\right)}\right\}$. It follows that the trace and the determinant of $f_{i_{12}}$ are zero. The second statement is the direct outcome of the fact that $f_{i_{12}}$ must verify its characteristic equation. □

Corollary 1 *When it exists, the center of rotation is the point at which $tr\left(f_{i_{12}} f_{i_{12}}^T\right)$ reaches a minimum zero value.*

Proof. The proof is straightforward from the second statement of the above theorem and the requirement that $f_{i_{12}}$ becomes nil at $Ci \equiv C_r$. □

The above development was based on the flexibility matrix. When a center of rotation exists, K_r is also block diagonal and it can be shown that we have

$$K_r = \begin{pmatrix} k_{r_{11}} & 0 \\ 0 & k_{r_{22}} \end{pmatrix} = \begin{pmatrix} f_{r_{11}}^{-1} & 0 \\ 0 & k_{r_{22}^{-1}} \end{pmatrix}. \quad (31)$$

And equivalent results can be derived based on the second Equation of the stiffness transformation given in Equation 30:

$$T = k_{i_{11}}^{-1} k_{i_{12}} = -k_{i_{12}}^{T} k_{i_{11}}^{-1}. \qquad (32)$$

It can be shown that this is the same matrix established in Equation 30, using the flexibility argument (use the inverse formula for a two-by-two block matrix for the derivation).

Finally, we close this Section by stating certain properties of a center of rotation. The center of rotation of a constrained structure has several characteristics. It is (a) a property of the geometry of the structure, (b) not necessarily connected to the flexible structure, (c) not necessarily coincident with the center of mass, (d) independent of the elastic properties of the structure, (e) independent of the mass properties of the structure, and (f) independent of the magnitude and direction of loads.

5. MEASURES FOR STRUCTURAL SYSTEMS

This section introduces three measures to assess the translation-rotation coupling, overall translation-translation flexibility and rotation-rotation stiffness. These measures help a designer compare different structural layouts and options. We start by introducing the concept of *coupling strength* between translation and rotation. This can give a designer the possibility to minimize the effect of parasitic motions in structures where coupling is unavoidable. There are three possibilities that one can immediately consider: Either using the coupling flexibility matrix $f_{i_{12}}$, or using the stiffness coupling matrix $k_{i_{12}}$, or a combination of the two. The following definitions introduce the three possibilities

Definition 4 *The translation-rotation* **flexibility coupling strength** *at a point C_i is defined as $\mathcal{A}_{f_i} = \sqrt{tr(f_{i_{12}} f_{i_{12}}^T)}$, where $f_{i_{12}}$ is the translation-rotation coupling flexibility matrix at that point.*

Definition 5 *The translation-rotation* **stiffness coupling strength** *at a point C_i is defined as $\mathcal{A}_{k_i} = \sqrt{tr(k_{i_{12}} k_{i_{12}}^T)}$, where $k_{i_{12}}$ is the translation-rotation coupling stiffness matrix at that point.*

Definition 6 *The translation-rotation* **mixed coupling strength** *at a point C_i is defined as $\mathcal{A}_i = \sqrt{tr(f_{i_{12}} k_{i_{12}}^T)}$, where $f_{i_{12}}$ and $k_{i_{12}}$ are the respective translation-rotation flexibility and stiffness matrices at that point.*

It can be shown that the three measures of coupling strength are frame invariant and therefore they can be used as structural properties. Next, let us address the *ranslation flexibility* and *rotation stiffness*. With the variation of the coupling strength from one point to another in 3D space, the translational flexibility also vary. However, the rotational flexibility remains invariant as expressed in Equations 26. A measure of overall translational flexibility may be defined as following:

Definition 7 *The overall translation-translation flexibility at a point C_i is defined as $\mathcal{F}_i = tr(f_{i_{11}})$, where $f_{i_{11}}$ is the translation-translation flexibility matrix at that point.*

On the other hand, the translation-translation stiffness remains invariant under transvections. It is the rotation-rotation stiffness matrix that varies as expressed in the third Equation of 28. A measure of overall rotation-rotation stiffness may be defined as following:

Definition 8 *The overall rotation-rotation stiffness of a structure at a point C_i is defined as $\mathcal{G}_i = tr(k_{i_{22}})$, where $k_{i_{22}}$ is the rotation-rotation stiffness matrix at that point.*

Finally, it is important to notice that $\mathcal{F}_i$, and, $\mathcal{G}_i$ are also frame invariant, and therefore can be used as measures.

6. DIFFERENTIATION PROPERTIES

Several results can be established using the introduction of the skew symmetric matrix format of the position vector r. let us use the ceil symbol to move between vector and matrix notations:

$$r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \lceil T \rceil = \left\lceil \begin{pmatrix} 0 & r_z & -r_y \\ -r_z & 0 & r_x \\ r_y & -r_x & 0 \end{pmatrix} \right\rceil \qquad (33)$$

The ceil operator has three key properties that that can be immediately extracted. Given two arbitrary skew symmetric matrices A and B, an arbitrary scalar λ and a vector r. Using the notation of Equation 33, we have the following:

$$\begin{array}{ll} \lceil A + B \rceil = \lceil A \rceil + \lceil B \rceil & \text{additivity} \\ \lceil \lambda A \rceil = \lambda \lceil A \rceil & \text{homogeneity} \\ A\, r = tr(A)\, r + \left\lceil (TA)^T - TA \right\rceil & \text{decomposition.} \end{array}$$

The first two properties make the ceil operator a *linear map* and the third property decomposes the effect of a matrix A on a particular vector into a colinear (axial) component along r and a deviatoric component. Using the fact that A can be written as the sum of a symmetric component A_s and a skewsymmetric component A_w, i.e.

$$A = A_s + A_w, A_s = \frac{A + A^T}{2} \text{ and } A_w = \frac{A - A^T}{2}. \qquad (34)$$

The deviatoric component of the decomposition property can be written as

$$\left\lceil (TA)^T - TA \right\rceil = -2 \lceil (TA)_w \rceil = 2 \left\lceil \left(A^T T\right)_w \right\rceil. \qquad (35)$$

That is, the deviatoric component is derived from a skewsymmetric matrix. In addition, it can be shown that the action of a skewsymmetric matrix results always in a nill axial component, while the deviatoric component of such action is orthogonal to the vector. Next, let A,B be two arbitrary matrices, the subscript s means the symmetric part and the subscript w refers to the skewsymmetric part. The following important differentiation rules hold:

$$\frac{\partial(tr(A))}{\partial T} = 0 \tag{36}$$

$$\begin{aligned} \frac{\partial(tr(AT))}{\partial T} &= \frac{\partial(tr(TA))}{\partial T} = -\frac{\partial(tr(A^T T))}{\partial T} \\ &= \lceil A^T - A \rceil = -2 \lceil A_w \rceil \end{aligned} \tag{37}$$

$$\frac{\partial(tr(ATB))}{\partial T} = \left\lceil (BA)^T - BA \right\rceil = -2 \lceil (BA)_w \rceil \tag{38}$$

$$\begin{aligned} \frac{\partial(tr(TAT))}{\partial T} &= -2 \lceil A_s^T T + T A_s \rceil = -4 \lceil (A_s T)_w \rceil \\ &= (A_s - tr(A_s) I_3) \lceil T \rceil \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{\partial \left(tr \left(A T^2 B \right) \right)}{\partial T} &= -2 \left\lceil (BA)_s^T T + T (BA)_s \right\rceil \\ &= 2 \left((BA)_s - tr (BA)_s I_3 \right) \lceil T \rceil \\ &= 4 \left\lceil \left((BA)_s T \right)_w \right\rceil \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{\partial \left(tr \left(A T B T \right) \right)}{\partial T} &= \frac{\partial \left(tr \left(T A T B \right) \right)}{\partial T} \\ &= -2 \lceil (ATB + BTA)_w \rceil \end{aligned} \tag{41}$$

If both A and B are symmetric matrices, then we have

$$\begin{aligned} \frac{\partial \left(tr \left(A T B T \right) \right)}{\partial T} &= \frac{\partial \left(tr \left(T A T B \right) \right)}{\partial T} \\ &= -2 \lceil ATB + BTA \rceil \end{aligned} \tag{42}$$

As a consequence of Equation 39, a new definition of the position vector r becomes readily available by substituting the particular case of $A = I_3$:

$$\begin{aligned} r &= \lceil T \rceil = -\frac{1}{4} \frac{\partial \left(tr(T^2) \right)}{\partial T} \\ &= \sqrt{-\frac{1}{2} tr\left(T^2\right)} \; \frac{\partial \left(\sqrt{-\frac{1}{2} tr(T^2)} \right)}{\partial T}. \end{aligned} \tag{43}$$

The most right hand side of the above Equation is a representation of r in the form of a modulus and a unit vector

$$\|r\| = \sqrt{-\frac{1}{2} tr\left(T^2\right)}, \; \frac{r}{\|r\|} = \frac{\partial \left(\sqrt{-\frac{1}{2} tr(T^2)} \right)}{\partial T}. \tag{44}$$

This additional representation of the direction of the variable position r in a gradient form is interesting and can be further exploited. Equation 39 indicates that the action of the symmetric component of an arbitrary matrix A on a vector r can be expressed as

$$A_s \, r = tr(A_s) \; r + \frac{1}{2} \frac{\partial(tr(TAT))}{\partial T}, \tag{45}$$

and therefore we can finally write

$$A \, r = tr(A) \; r + A_w \; r + \frac{1}{2} \frac{\partial(tr(TAT))}{\partial T}. \tag{46}$$

Obviously, this decomposition shows a quadratic term which indicates the existence of stationary points with physical implications. In addition to the above differentiation rules, we will make use of the following additional differentiation formula in deriving two stationary points pertaining to translation and rotation energy storage in the following Section **??**. let us write the matrix T as a linear combination of three skew symmetric permutations:

$$T = r_x Q_x + r_y Q_y + r_z Q_z, \tag{47}$$

where

$$Q_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; Q_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{and } Q_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily shown that for any arbitrary matrix A and for any component $i = x, y$ or z, we have the following chain rule

$$\frac{\partial \, (.)}{\partial r_i} = \frac{\partial \, (.)}{\partial T} \frac{\partial T}{\partial r_i} = \frac{\partial \, (.)}{\partial T} Q_i \tag{48}$$

It can be shown that we immediately have the following identities:

$$\frac{\partial \, (A \, T)}{\partial r_i} = A Q_i \, , \; \frac{\partial \, (T \, A)}{\partial r_i} = Q_i A$$

$$\text{and } \frac{\partial \, (T \, A \, T)}{\partial r_i} = T \, A \, Q_i + Q_i \, A \, T \, . \tag{49}$$

7. STATIONARY POINTS IN STRUCTURES

We start by addressing the first three definitions of Section 5 for minimum coupling strength between degrees of freedom. Three corresponding points are found. Subsequently, the last two definitions of Section 5 on minimum translation flexibility and rotational stiffness are used to derive the locations of the two corresponding points. Finally, it is shown that when a center of rotation exists, These points coalesce with the points of minimum coupling (flexibility, stiffness and mixed flexibility-stiffness), and points of minimum translation flexibility and maximum rotation stiffness.

7.1. Point of minimum flexibility coupling strength

The point in 3D space where translation-rotation flexibility coupling is at a minimum is obtained from an arbitrary point C_i using the result of the following theorem:

Theorem 3 *Any structure with proper constraints has a point in 3D space, C_o at which the translation-rotation coupling strength is absolute minimum. This point is obtained from an arbitrary point C_i using:*

$$\overrightarrow{C_iC_o} = \left(f_{i22}^2 - tr(f_{i22}^2)I_3\right)^{-1}\lceil f_{i12}f_{i22} - f_{i22}f_{i12}^T\rceil \quad (50)$$

This point is at which the deviatoric component of the action of f_{i22}^2 on $\overrightarrow{C_iC_o}$ is twice the ceil of the skewsymmetric part of $f_{i12}f_{i22}$, and is solution of the partial differential equation:

$$\frac{\partial(tr(\frac{1}{2}Tf_{i22}^2T + f_{i12}f_{i22}T))}{\partial T} = 0. \quad (51)$$

Proof. Using Definition 4, and substituting the coupling matrix expression from the second Equation in 26, which means the following expressions

$$\frac{\partial}{\partial T}\left(\sqrt{tr(f_{o_{12}}f_{o_{12}}^T)}\right),$$
$$\frac{\partial}{\partial T}\left(\sqrt{tr\left((f_{i_{12}} + Tf_{i_{22}})(f_{i_{12}}^T - f_{i_{22}}T)\right)}\right), \text{ and}$$
$$\frac{\partial}{\partial T}\left(tr\left(f_{i_{12}}f_{i_{12}}^T - f_{i_{12}}f_{i_{22}}T + Tf_{i_{22}}f_{i_{12}}^T - Tf_{i_{22}}^2T\right)\right),$$

are nil. Using the differentiation rules established in Equations 36, 37 and Equation 39, and carrying the algebra, we obtain the results of the theorem. □

It is interesting to note that the point of minimal flexibility coupling is a symmetry point as indicated by the result of the following corollary.

Corollary 2 *The point of minimum flexibility coupling strength makes the matrix $(Tf_{i22} + f_{i12})\,f_{i22}$ take a symmetric form. When $(Tf_{i22} + f_{i12})\,f_{i22} = 0$, such point is a center of rotation.*

Proof. Equation 51 can be rewritten as

$$\left(f_{i22}^2 - tr(f_{i22}^2)I_3\right)\lceil T\rceil - \lceil f_{i12}f_{i22} - f_{i22}f_{i12}^T\rceil = 0.$$

Using the most righthand side equality in Equation 39, we have

$$-2\left\lceil\left(f_{i22}^2T\right)_w\right\rceil - \lceil f_{i12}f_{i22} - f_{i22}f_{i12}^T\rceil =$$
$$-2\left\lceil\left(Tf_{i22}^2\right)_w + (f_{i12}f_{i22})_w\right\rceil = 0,$$

after making use of the additivity property of the ceil operator. The most right equality is then simplified to be $\lceil((Tf_{i22} + f_{i12})\,f_{i22})_w\rceil = 0$. That is $(Tf_{i22} + f_{i12})\,f_{i22}$ is symmetric. The second statement of the corollary is immediately shown when writing $(Tf_{i22} + f_{i12})\,f_{i22} = 0$. We have to have $Tf_{i22} + f_{i12} = 0$, the necessary and sufficient condition for a center of rotation to exist as established by Equation 30. □

Next, we show that any structure possess a particular point in 3D space where the translation-translation flexibility reaches a minimum and a third point where the rotation-rotation stiffness reaches maximum.

7.2. Point of minimum stiffness coupling strength

The point in 3D space where translation-rotation stiffness coupling is at a minimum is obtained from an arbitrary point C_i using the result of the following theorem:

Theorem 4 *Any structure with proper constraints has a point in 3D space, C_s at which the translation-rotation coupling strength is absolute minimum. This point is obtained from an arbitrary point C_i using:*

$$\overrightarrow{C_iC_s} = \left(k_{i11}^2 - tr(k_{i11}^2)I_3\right)^{-1}\lceil -k_{i11}k_{i12} + k_{i12}^Tk_{i11}\rceil. \quad (52)$$

This point is at which the deviatoric component of the action of k_{i11}^2 on $\overrightarrow{C_iC_s}$ is twice the ceil of the skewsymmetric part of $k_{i12}^Tk_{i11}$, and is solution to the partial differential equation:

$$\frac{\partial(tr(\frac{1}{2}Tk_{i11}^2T + k_{i11}k_{i12}T))}{\partial T} = 0. \quad (53)$$

Proof. Using Definition 5, and substituting the coupling matrix expression from the second Equation in lemma 2, we obtain equivalent expressions to those in the previous theorem. Using the differentiation rules established in Equations 36, 37 and Equation 39, and carrying the algebra, we obtain the results of the theorem. □

7.3. Point of minimum mixed coupling strength

The point in 3D space where coupling strength in a mixed stiffness-flexibility sense is at a minimum is obtained from an arbitrary point C_i using the result of the following theorem:

Theorem 5 *Any structure with proper constraints has a unique point in 3D space, C_q at which mixed coupling strength is absolute minimum. This point is obtained from an arbitrary point C_i using:*

$$\begin{aligned}\overrightarrow{C_iC_q} = -(&\ (tr(f_{i22}k_{i11}) - tr(f_{i22})tr(k_{i11}))I_3 \\ &-(f_{i22} - tr(f_{i22})I_3)k_{i11} \\ &-(k_{i11} - tr(k_{i11})I_3)f_{i22})^{-1}\lceil k_{i11}f_{i12} + f_{i22}k_{i12}^T\rceil.\end{aligned} \quad (54)$$

Proof. Using Definition 6, and substituting the flexibility and stiffness coupling matrices from the second Equation in 26, and the second Equation of 28, the following terms are nil:

$$\frac{\partial}{\partial T}\left(\sqrt{tr(f_{o_{12}}k_{o_{12}}^T)}\right),$$
$$\frac{\partial}{\partial T}\left(\sqrt{tr\left((f_{i_{12}} + Tf_{i_{22}})(k_{i_{12}}^T + Tk_{i_{11}})\right)}\right),$$
$$\frac{\partial}{\partial T}\left(tr\left(f_{i_{12}}k_{i_{12}}^T + f_{i_{12}}Tk_{i_{11}} + Tf_{i_{22}}k_{i_{12}}^T + Tf_{i_{22}}Tk_{i_{11}}\right)\right).$$

Using the differentiation rules established in Equations 36, 37 and Equation 42, and carrying the algebra, we obtain the results of the theorem. □

7.4. Points of minimum translation flexibility and maximum rotation stiffness

For flexibility and stiffness, we have two stationary points: point of minimum translation flexibility and point of maximum rotation stiffness. These points are obtained as following:

Theorem 6 *Any arbitrarily structure with proper constraints possess a point in 3D space C_t at which the overall translation-translation flexibility is minimum and is obtained from an arbitrary point C_i using*

$$\overrightarrow{C_iC_t} = \left(f_{i_{22}} - tr\left(f_{i_{22}}\right) I_3\right)^{-1} \left\lceil f_{i12} - f_{i12}^T \right\rceil . \quad (55)$$

This point is at which the deviatoric component of the action of f_{i22} on $\overrightarrow{C_iC_t}$ is twice the ceil of the skewsymmetric part of f_{i12}, and solution of:

$$\frac{\partial(tr(\frac{1}{2}Tf_{i22}T - f_{i12}T))}{\partial T} = 0. \quad (56)$$

Proof. Following the same procedure as in the proof of the result of Theorem 3, using Definition 7, we write that the trace of the right hand side of the first Equation of 26 is stationary. This leads to:

$$\frac{\partial}{\partial T}\left(tr(f_{i_{11}} + \left(Tf_{i_{12}}^T + f_{i_{12}}T^T\right) + T^T f_{i_{22}}T)\right) = 0. \quad (57)$$

Applying the identities given in Equations 36 through 39, and carrying the algebra leads to the results of the theorem. □

Rotation stiffness is important to numerous applications, especially those that can be found in mounting optics and in pointing and tracking. Such point can be established using the following theorem:

Theorem 7 *Any arbitrarily structure with proper constraints possess a point in 3D space, C_m at which the rotation stiffness is maximum; such point is obtained from an arbitrary point C_i using*

$$\overrightarrow{C_iC_m} = -\left(k_{i_{11}} - tr\left(k_{i_{11}}\right) I_3\right)^{-1} \left\lceil k_{i12} - k_{i12}^T \right\rceil . \quad (58)$$

This point is at which the deviatoric component of the action of k_{i22} on $\overrightarrow{C_iC_t}$ is minus two times the ceil of the skewsymmetric part of k_{i12}, and is solution to the partial differential equation:

$$\frac{\partial(tr(-\frac{1}{2}Tk_{i11}T - k_{i12}^T T))}{\partial T} = 0. \quad (59)$$

Proof. Following the same procedure as in the proof of the result of Theorem 6, using Definition 7, we write that the trace of the right hand side of the last Equation of 28 is stationary. This leads to:

$$\frac{\partial}{\partial T}\left(tr(k_{i_{22}} + \left(Tk_{i_{12}} - k_{i_{12}}^T T\right) - T^T k_{i_{11}}T)\right) = 0. \quad (60)$$

Applying the identities given in Equations 36 through 39, and carrying the algebra leads to the results of the theorem. □

7.5. Point of minimum strain energy storage

Consider an arbitrary flexible structure with proper constraints. There exists a point in 3D space through which the application of external loads results in absolute minimum strain energy storage due to translation.

Theorem 8 *An arbitrary structure with proper constrains has an associated point through which external loads result in minimum translation strain energy storage. This point coincides with the point of minimum translation flexibility.*

Proof. Consider the translation-rotation partition. The strain energy of a flexible system under the application of a load at point C_r can be written as a sum of translation-translation, rotation-rotation and mixed translation-rotation contributions,

$$E_r = E_{1r} + E_{2r} + E_{12r}, \quad (61)$$

where the strain energy components, using the flexibilities, are given by

$$E_{1r} = \frac{1}{2}P_{r_1}^T f_{r_{11}} P_{r_1},\ E_{2r} = \frac{1}{2}P_{r_2}^T f_{r_{22}} P_{r_2},\ \text{and}$$
$$E_{12r} = P_{r_1}^T f_{r_{12}} P_{r_2}. \quad (62)$$

Substituting the first Equation of 29, in the expression of E_{1r} leads to

$$E_{1r} = \frac{1}{2}P_{r_1}^T f_{i_{11}} P_{r_1} +$$
$$\frac{1}{2}P_{r_1}^T \left(Tf_{i_{12}}^T + f_{i_{12}}T^T + T^T f_{i_{22}}T\right) P_{r_1}. \quad (63)$$

Differentiating with respect to r_i using the identities given in 49, and simplifying leads to

$$\frac{\partial E_{1_r}}{\partial r_i} = P_{i_1}^T Q_i \left(f_{i_{12}}^T - f_{i_{22}}T\right) P_{i_1}. \quad (64)$$

and therefore E_{1r} is stationary when we have

$$T = f_{i_{22}}^{-1} f_{i_{12}}^T. \quad (65)$$

Differentiating Equation 64 with respect to r_i leads to

$$\frac{\partial^2 E_{1_r}}{\partial r_i^2} = -P_{i_1}^T Q_i f_{i_{22}} Q_i P_{i_1} > 0. \quad (66)$$

It can be shown that the above term is positive because it is the strain energy stored in the structure due to the force indirectly normal to projection of P_{i1} on the plane (j, k). This means that we have a stationary minimum rotation storage point which happen to correspond to the center of rotation, as the above expression for T is the one given by Equation 30. Such statement assumes that the structure possesses a center of rotation. If the center of rotation does not exist, then E_{1r} is a monotonic function of T which upon using the decomposition identity 46, we have

$$E_{1r} = \frac{1}{2}P_{r_1}^T tr(f_{r_{11}})P_{r_1} + \frac{1}{4}P_{r_1}^T \frac{\partial(tr(Tf_{r_{11}}T))}{\partial T}. \quad (67)$$

The term $\frac{1}{2}P_{r_1}^T tr(f_{r_{11}})P_{r_1}$ is already known to be minimum at the point of minimum translation flexibility C_t per Theorem 6, and therefore E_{1r} is minimum. □

Similar to the minimum translation strain energy storage development of the previous section, we show that the rotation energy storage also possess stationary points. There exists a point through which the resultant external loads are applied, the strain energy due to rotation is also at absolute minimum.

Theorem 9 *An arbitrary structure with proper constrains has an associated point through which external loads result in minimum rotation strain energy storage. This point is also the point of maximum rotation stiffness*

Proof. Again, consider the translation-rotation partition. The strain energy of a flexible system under the application of a load at an arbitrary point C_r can be written as a sum of translation-translation, rotation-rotation and mixed translation-rotation contributions, and therefore we can write

$$Ek_r = Ek_{1r} + Ek_{2r} + Ek_{12r} \tag{68}$$

Using the stiffness matrix, the strain energy components are given by

$$Ek_{1r} = \frac{1}{2}u_{r_1}^T k_{r_{11}} u_{r_1},\ Ek_{2r} = \frac{1}{2}u_{r_2}^T k_{r_{22}} u_{r_2},\ \text{and}$$
$$Ek_{12r} = u_{r_1}^T k_{r_{12}} u_{r_2}. \tag{69}$$

Substituting the third Equation of 28, in the expression of Ek_{2r} leads to

$$Ek_{2r} = \frac{1}{2}u_{r_2}^T\, k_{i_{22}} u_{r_2} +$$
$$\frac{1}{2}u_{r_2}^T \left(Tk_{i_{12}} - k_{i_{12}}^T T + T^T k_{i_{11}} T\right) u_{r_2}. \tag{70}$$

Differentiating with respect to r_i using Equations 49, and simplifying leads to

$$\frac{\partial Ek_{2_r}}{\partial r_i} = -u_{i_2}^T Q_i \left(k_{i_{11}} T - k_{i_{12}}\right) u_{i_2}. \tag{71}$$

In the above expression , we also made use of the fact $u_{r2} = u_{i2}$. It can be then stated that such derivative is nil for either r_x, r_y, or r_z when $T = k_{i_{11}}^{-1} k_{i_{12}}^T$ for any u_{i2}. Differentiating again with respect to r_i leads to

$$\frac{\partial^2 Ek_{2_r}}{\partial r_i^2} = -u_{i_2}^T Q_i k_{i_{11}} Q_i u_{i_2} > 0. \tag{72}$$

This means that we have a stationary minimum rotation storage point which happen to correspond to the center of rotation, should such point exists, because the above expression for T is the one given by Equation 32. If the center of rotation does not exist, then Ek_{2r} is either a monotonically increasing or monotonically decreasing function of T. And if we substitute the decomposition identity 46, we have

$$Ek_{2r} = \frac{1}{2}u_{r_1}^T k_{r_{11}} u_{r_1} =$$
$$\frac{1}{2}u_{r_1}^T tr(k_{r_{11}}) u_{r_1} + \frac{1}{4}u_{r_1}^T \frac{\partial(tr(Tk_{r_{11}}T))}{\partial T} \quad . \tag{73}$$

The term $\frac{1}{2}u_{r_1}^T tr(k_{r_{22}})u_{r_2}$ is minimum at the point of maximum rotation stiffness C_m per Theorem 7. □

7.6. Relationship between critical points

Another important result is immediately deduced and documented in the following theorem:

Theorem 10 *When exists, a center of rotation coincides with the point of minimum coupling strength, the point of minimum translation flexibility, the point of maximum rotation stiffness, the point of minimum translation energy storage and the point of minimum rotation strain energy storage.*

Proof. Suppose the initial point C_i coincides with the center of rotation, then by definition, we have to have $f_{i12} = 0$. Using Theorem 3 and Theorem 6, $\overrightarrow{C_iC_o} = \overrightarrow{C_iC_t} = 0$. It can be shown that $k_{i_{12}} = 0$ by expressing that the the six-dof block stiffness matrix F_r is the inverse of the flexibility matrix F_i in Equations 25. Substituting $k_{i_{12}} = 0$ in Equation 58 leads to $\overrightarrow{C_iC_m} = 0$. □

8. EXAMPLES OF C_R DETERMINATION

8.1. Center of Rotation of U-shaped flexures

As an example, consider the four inverted U-shaped flexures used to mount a fast steering mirror shown in figure 3. The outer straight edges of the mirror are clamped to the static side (ground) and the inner opposite straight edges are attached to the steering subassembly, such that only the dome part flexes. The inertial loads of the Mirror and the remaining mirror mount hardware need to be located on the Center of rotation which is expected to be on the z-axis because of the double symmetry of the layout. The point C_i is placed intentionally off-axis at coordinates $(0, 5, 5)$. The output of the flexibility matrix F_i and computation of the matrix T using Equation (30) leads to

```
 3.19E-3 -8.09E-5 -1.02E-2  4.45E-7  2.04E-3 -1.62E-5
-8.09E-5  3.19E-3 -1.02E-2 -2.04E-3 -2.99E-7  1.62E-5
-1.02E-2 -1.02E-2  7.80E-2  6.90E-3 -6.90E-3 -1.69E-8
 4.45E-7 -2.04E-3  6.90E-3  1.38E-3  5.91E-7 -1.55E-9
 2.04E-3 -2.99E-7 -6.90E-3  5.91E-7  1.38E-3 -7.29E-9
-1.62E-5  1.62E-5 -1.69E-8 -1.55E-9 -7.29E-9  3.24E-6
```

$$T = \begin{pmatrix} 0.31758E-3 & -0.14812E+1 & 0.50000E+1 \\ 0.14812E+1 & -0.44397E-3 & 0.50007E+1 \\ -0.49999E+1 & 0.50000E+1 & 0.14079E-1 \end{pmatrix}$$

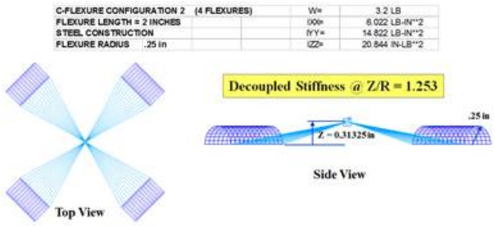

Figure 3. Finite Element model of a fast steering mirror flexures with a rigid body simulating the mirror subassembly

Verification of the results through a Nastran run by moving C_i and placing it right on C_r through placing now the master node (Grid 3) of the RBE2 at $(5, 5, 0) + (-5, -5, -1.4812) = (0, 0, -1.4812)$ leads to the following flexibility matrix

7.87E-5	-2.43E-8	-6.06E-8	-4.38E-7	-3.24E-9	-16.9E-9
-2.43E-8	7.87E-5	1.00E-7	8.38E-8	6.13E-7	2.29E-9
-6.06E-8	1.00E-7	8.97E-3	-5.05E-8	-9.38E-9	-4.56E-8
-4.38E-7	8.38E-8	-5.05E-8	1.38E-3	5.91E-7	-1.55E-9
-3.24E-9	6.13E-7	-9.38E-9	5.91E-7	1.38E-3	-7.29E-9
-16.9E-9	2.29E-9	-4.56E-8	-1.55E-9	-7.29E-9	3.24E-6

The diagonal flexibility matrix is

9.E-5 7.8E-5 9.E-3 1.38E-3 1.38E-3 3.24E-6.

These values are identical to the diagonal terms of the above flexibility computed matrix.

8.2. Pyramid flexure -Pure Center of Rotation:

The flexure is held at two legs in the x-z plane. The other two legs in the y-z plane are rigidly attached to a point located at $(x, y, z) = (0, 0, 0.63255)$. The output of the flexibility matrix f_i and computation of the matrix T using Equation (30) leads to the following:

2.86E-3	1.99E-9	-7.62E-8	4.48E-7	3.01E-2	2.80E-8
1.99E-9	2.86E-3	8.36E-8	-3.01E-2	-3.80E-7	1.36E-8
-7.62E-8	8.36E-8	3.71E-3	-2.25E-6	-2.35E-6	-3.43E-9
4.48E-7	-3.01E-2	-2.25E-6	4.76E-1	1.74E-5	-3.53E-7
3.01E-2	-3.80E-7	-2.35E-6	1.74E-5	4.76E-1	9.49E-9
2.80E-8	1.36E-8	-3.43E-9	-3.53E-7	9.49E-9	4.89E-3

$$T = \begin{pmatrix} 0.32E-5 & -0.63E-1 & -0.56E-5 \\ 0.63E-1 & -0.15E-5 & 0.18E-5 \\ 0.47E-5 & 0.49E-5 & 0.70E-6 \end{pmatrix}$$

Placement of the center of rotation at the following coordinates $(2.83E-7, -4.71E-9, -.063256)$ Leads to the decoupled flexibility matrix shown below:

9.50E-4	-1.54E-8	7.25E-8	-6.55E-7	5.95E-7	2.74E-8
-1.54E-8	9.50E-4	-5.90E-8	-6.43E-7	7.23E-7	-8.69E-9
7.25E-8	-5.90E-8	3.71E-3	-2.25E-6	-2.35E-6	-3.43E-9
-6.55E-7	-6.43E-7	-2.25E-6	4.76E-1	1.74E-5	-3.53E-7
5.95E-7	7.23E-7	-2.35E-6	1.74E-5	4.76E-1	9.48E-9
2.74E-8	-8.69E-9	-3.43E-9	-3.53E-7	9.48E-9	4.89E-3

8.3. Example of Shear Center

Perhaps the most referenced problem in studying the shear center in stress analysis is the cantilever bracket cross section (see Figure 8.2). The flexibility matrix of this beam is given at an arbitrary point C_i

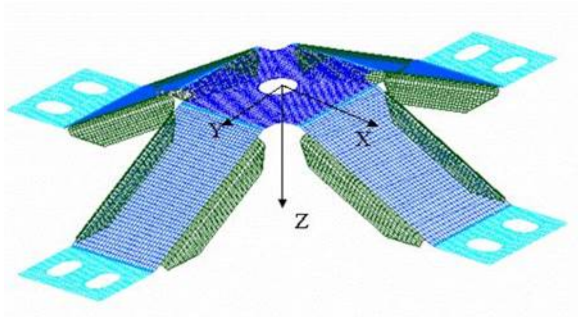

Figure 4. Pyramid Flexure

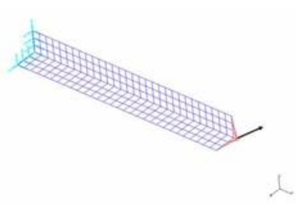

Figure 5. Bracket Cantilever Beam

6.65E-6	-5.65E-5	-5.65E-5	1.46E-16	3.32E-6	-3.32E-6
-5.65E-5	3.68E-3	-2.06E-3	-1.78E-3	-2.83E-5	4.71E-5
-5.65E-5	-2.06E-3	3.68E-3	1.78E-3	-4.71E-5	2.83E-5
1.46E-16	-1.78E-3	1.78E-3	1.19E-3	7.22E-17	-1.24E-16
3.32E-6	-2.83E-5	-4.71E-5	7.22E-17	2.77E-6	-1.66E-6
-3.32E-6	4.71E-5	2.83E-5	-1.24E-16	-1.66E-6	2.77E-6

and the matrix T can be readily computed to be

$$T = \begin{pmatrix} 0.43344E-15 & -0.75001E+0 & 0.75001E+0 \\ 0.14994E+1 & -0.18641E-5 & -0.17000E+2 \\ -0.14994E+1 & 0.17000E+2 & -0.24484E-6 \end{pmatrix}$$

9. JITTER LOADS FROM MECHANISMS

Consider the soft spring-mounted structure shown in Figure 9. This configuration covers a number of mechanisms which are driven by actuators while mounted on a flexural system, which in-turn attaches to a rigid mount. Three encoders are adequately positioned to deliver the tip, tilt and pogo motions of the payload, i.e. θ_x, θ_y, and u_z to a control system operating in a close loop configuration. Without loss of generality, we assume that the flexural system possess radial symmetry, i.e. $k_x = k_y = k_l$ and $k_{xx} = k_{yy} = k_{ll}$. We further assume thats the payload is relatively rigid with a mass m and double inertial symmetry such that the cross-inertial terms I_{xz} and I_{yz} are nil. The flexural system is such that it possesses a center of rotation C_R, intended to be coincident with the center of mass of the payload C_G. Despite best efforts, tolerances in manufacturing and assembly will inevitably produce an offset $[r_x, r_y, r_z]$ between C_G and C_R. The performance of a precision mechanism will depend on how well the offsets are minimized, and therefore a post-manufacturing plan needs to be in place to balance the mechanism in three directions. Shimming, match machining and/or post-machining are often used in the final

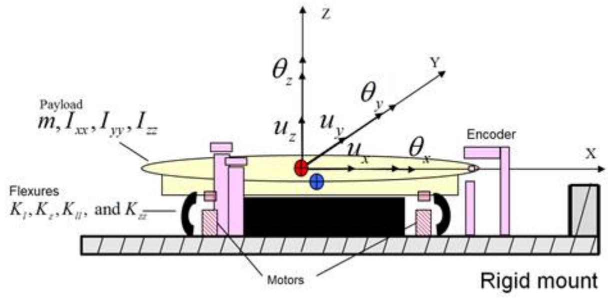

Figure 6. Soft spring-mounted structure

stages of the assembly. For complex multi-component systems, inertial balancing follows the process described in [6] where rotary tables can be used for non spinning mechanisms. The $C_G - C_R$ offset verification process follows the methodology presented in [7] where the mechanism is placed on a modular force measurement table tuned to have its modes outside the natural frequencies of the system.

Let us denote by M the lumped mass diagonal 6 by 6 matrix of the payload, derived through static condensation at point C_G or even through a standard mass properties generation procedure. Let us also denote by K_R the diagonal stiffness matrix produced at the center of rotation C_R, the eigenvectors and eigenvalues of the fundamental modes of the mechanism can be obtained from the matrix $M^{-1}K_R$, sum of $M^{-1}K_G$, representing the target design:

$$\mathrm{diag}\begin{pmatrix} w_l^2 & w_l^2 & w_z^2 & w_{xx}^2 & w_{yy}^2 & w_{zz}^2 \end{pmatrix}$$

where $w_l^2 = k_l/m$, $w_z^2 = k_z/m$, $w_{xx}^2 = k_{ll}/I_{xx}$, $w_{yy}^2 = k_{yy}/I_{yy}$, $w_{zz}^2 = k_{zz}/I_{zz}$ and the matrix representing the combined effect of tolerances and other errors, given by

$$\begin{matrix}
0 & 0 & 0 & 0 & r_z w_l^2 & -r_y w_l^2 \\
0 & 0 & 0 & -r_z w_l^2 & 0 & r_x w_l^2 \\
0 & 0 & 0 & r_y w_z^2 & -r_x w_z^2 & 0 \\
0 & -\frac{r_z}{t_x^2} w_l^2 & \frac{r_y}{t_x^2} w_z^2 & \frac{r_y^2}{t_x^2} w_z^2 + \frac{r_z^2}{t_x^2} w_l^2 & -\frac{r_x r_y}{t_x^2} w_z^2 & -\frac{r_x r_z}{t_x^2} w_l^2 \\
\frac{r_z}{t_y^2} w_l^2 & 0 & -\frac{r_x}{t_y^2} w_z^2 & -\frac{r_x r_y}{t_y^2} w_z^2 & \frac{r_x^2}{t_y^2} w_z^2 + \frac{r_z^2}{t_y^2} w_l^2 & -\frac{r_y r_z}{t_y^2} w_l^2 \\
-\frac{r_y}{t_z^2} w_l^2 & \frac{r_x}{t_z^2} w_l^2 & 0 & -\frac{r_x r_z}{t_z^2} w_l^2 & -\frac{r_y r_z}{t_z^2} w_l^2 & \frac{r_x^2 + r_y^2}{t_z^2} w_l^2
\end{matrix}$$

where $t_x^2 = Ixx/m, t_y^2 = Iyy/m,$ and $t_z^2 = Izz/m$ are the gyration radii. Multivariate Taylor series expansions of the fundamental modes around the target mechanism design parameters are given by, for small offsets $[r_x, r_y, rz]$

$$\begin{matrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & \frac{w_l^2}{w_l^2 - w_{xx}^2} r_z & -\frac{w_z^2}{w_z^2 - w_{xx}^2} r_y \\
-\frac{w_l^2}{w_l^2 - w_{yy}^2} r_z & 0 & \frac{w_z^2}{w_z^2 - w_{yy}^2} r_x \\
\frac{w_l^2}{w_l^2 - w_{zz}^2} r_y & -\frac{w_l^2}{w_l^2 - w_{zz}^2} r_x & 0
\end{matrix}$$

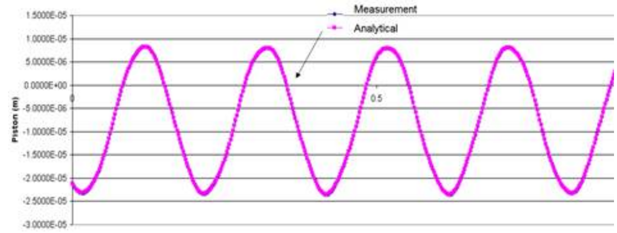

Figure 7. Axial motion -analysis versus measurements

$$\begin{matrix}
0 & \frac{r_z w_l^2}{t_y^2 (w_l^2 - w_{yy}^2)} \\
-\frac{r_z w_l^2}{t_x^2 (w_l^2 - w_{xx}^2)} & 0 \\
\frac{r_y w_z^2}{t_x^2 (w_z^2 - w_{xx}^2)} & -\frac{r_x w_z^2}{t_y^2 (w_z^2 - w_{yy}^2)} \\
1 + \frac{w_{xx}^2 w_z^2 r_y^2}{t_x^2 (w_z^2 - w_{xx}^2)^2} + \frac{w_{xx}^2 w_l^2 r_z^2}{t_x^2 (w_l^2 - w_{xx}^2)^2} & 0 \\
0 & 1 + \frac{w_{yy}^2 w_z^2 r_x^2}{t_y^2 (w_z^2 - w_{yy}^2)^2} + \frac{w_{yy}^2 w_l^2 r_z^2}{t_y^2 (w_l^2 - w_{yy}^2)^2} \\
0 & 0
\end{matrix}$$

$$\begin{matrix}
-\frac{w_l^2}{t_z^2 (w_l^2 - w_{zz}^2)} r_y \\
\frac{w_l^2}{t_z^2 (w_l^2 - w_{zz}^2)} r_x \\
0 \\
0 \\
0 \\
1 + \frac{w_{zz}^2 w_l^2}{t_z^2 (w_l^2 - w_{zz}^2)^2} (r_x^2 + r_y^2)
\end{matrix}$$

Taylor series expansions of the first six frequencies are

$$\begin{matrix}
w_l^2 + \frac{w_l^4}{t_z^2 (w_l^2 - w_z^2)} r_y^2 + \frac{w_l^4}{t_y^2 (w_l^2 - w_{yy}^2)} r_z^2 \\
w_l^2 + \frac{w_l^4}{t_z^2 (w_l^2 - w_z^2)} r_x^2 + \frac{w_l^4}{t_x^2 (w_l^2 - w_{xx}^2)} r_z^2 \\
w_z^2 + \frac{w_z^4}{t_y^2 (w_l^2 - w_z^2)} r_x^2 + \frac{w_z^4}{t_x^2 (w_l^2 - w_{yy}^2)} r_y^2 \\
w_{xx}^2 + \frac{w_{xx}^2 w_z^2}{t_x^2 (w_z^2 - w_{xx}^2)} r_y^2 - \frac{w_{xx}^2 w_l^2}{t_x^2 (w_l^2 - w_{xx}^2)} r_z^2 \\
w_{yy}^2 - \frac{w_{yy}^2 w_z^2}{t_y^2 (w_z^2 - w_{yy}^2)} r_x^2 - \frac{w_{yy}^2 w_l^2}{t_y^2 (w_l^2 - w_{yy}^2)} r_z^2 \\
w_{zz}^2 - \frac{w_{zz}^2 w_l^2}{t_z^2 (w_l^2 - w_{zz}^2)} r_x^2 - \frac{w_{zz}^2 w_l^2}{t_z^2 (w_l^2 - w_{zz}^2)} r_y^2
\end{matrix}$$

From the above results, we may state that (a) an offset r_z along the z axis causes coupling between tip/tilt modes and lateral modes, (b) in-plane offsets r_x or r_y cause coupling between tip/tilt and axial motions, (c) the leading expansion terms of the motions and forces are linear functions of offsets and (d) frequencies have no first order terms in offsets. Having derived the fundamental modes and frequencies of a non-perfect system, and having characterized the effect of offsets in the case of inertial loads, it remains to measure r_x, r_y and r_z from tests so that a payload location is adjusted and the mechanism is brought to accurate balance. The FSM can be driven by one of the following harmonic loads using its own actuators in open-loop:

$P_z = P_0 sin(\omega t)$, $M_x = M_0 sin(\omega t)$, and $M_y = M_0 sin(\omega t)$.

Measurement of the axial, tip and tilt motions of the mirror may be obtained using the FSM's own encoders if they were already

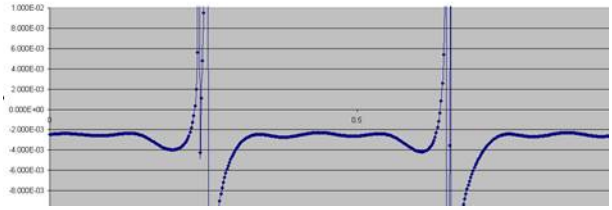

Figure 8. r_y measured to be 0.0025 inch (64 microns) in a steering mirror prior to adjustment

calibrated as part of the system or provided independently using a three-axes interferometer and the results are compared to the analytical expected motions. The respective six-dof time responses under unit harmonic loads are given by

$$\begin{matrix} 0 \\ 0 \\ \frac{\phi(w_z,\omega)}{m w_z} \\ \frac{w_z\eta(w_z,w_{xx},\omega)}{I_{xx}w_{xx}} r_y \\ \frac{w_z\eta(w_z,w_{yy},\omega)}{I_{yy}w_{yy}} r_x \\ 0 \end{matrix}, \quad \begin{matrix} 0 \\ \frac{w_y\eta(w_y,w_{xx},\omega)}{I_{xx}w_{xx}} r_z \\ -\frac{w_z\eta(w_z,w_{xx},\omega)}{I_{xx}w_{xx}} r_y \\ \frac{\phi(w_{xx},\omega)}{I_{xx}w_{xx}} \\ 0 \\ 0 \end{matrix}, \quad \begin{matrix} -\frac{w_x\eta(w_x,w_{yy},\omega)}{I_{yy}w_{yy}} r_z \\ 0 \\ \frac{w_z\eta(w_z,w_{yy},\omega)}{I_{yy}w_{yy}} r_x \\ 0 \\ \frac{\phi(w_{yy},\omega)}{I_{yy}w_{yy}} \\ 0 \end{matrix}.$$

And the corresponding reactions are

$$\begin{matrix} 0 \\ 0 \\ -\phi(w_z,\omega) \\ (w_z\eta(w_z,w_{xx},\omega)-\phi(w_z,\omega))r_y \\ (\phi(w_z,\omega)-w_z\eta(w_z,w_{xx},\omega))r_x \\ 0 \end{matrix}, \quad \begin{matrix} 0 \\ \frac{-k_y\mu(w_y,w_{xx},\omega)r_z}{I_{xx}} \\ \frac{k_z\mu(w_z,w_{xx},\omega)r_y}{I_{xx}} \\ -\phi(w_{xx},\omega) \\ 0 \\ 0 \end{matrix}, \quad \begin{matrix} \frac{k_x\mu(w_x,w_{yy},\omega)r_z}{I_{yy}} \\ 0 \\ \frac{-k_z\mu(w_z,w_{yy},\omega)r_x}{I_{yy}} \\ 0 \\ -\phi(w_{yy},\omega) \\ 0 \end{matrix}.$$

where

$$\phi(w_i,\omega) = \frac{w_i sin(\omega t) - \omega sin(w_i t)}{w_i^2 - \omega^2},$$

$$\eta(w_i,w_j,\omega) = \frac{w_i\phi(w_j,\omega) - w_j\phi(w_i,\omega)}{w_i^2 - w_j^2},$$

$$\mu(w_i,w_j,\omega) = \frac{w_j\phi(w_j,\omega) - w_i\phi(w_i,\omega)}{w_i^2 - w_j^2},$$

and indices i and$j$ correspond to the loading and response degrees of freedom. The expressions are restricted to mechanisms where the axial and bending frequencies are separate which is usually the case. The six reaction loads can be measured by placing the FSM on a force measurement table. From the analytical response, it is obvious that offsets can be obtained either from measured motions or measured reaction loads. Requirements are often specified in terms of motions and imparted reaction loads, and therefore provisions for requirement verification already dictate to development of the necessary test setups. This makes a balancing effort free of added cost burden as the same test setups are used for both requirement verification and balancing efforts. To illustrate a typical balancing effort, the mirror is driven at a frequency of about 5 Hz in the axial direction (Z-harmonic test). The intended pure z-motion is associated with unwanted tip and tilt as well parasitic jitter moments about x and y.Figure 7 shows the piston motion using current analytical solution to match measurements. Measurement of the y direction offset r_y is shown in Figure 8 prior to balancing. The value oscillates around a median value of 0.0025 inches indicating that the unit was already properly assembled. Following a quick lateral shimming adjustment, the value went down to 0.0013 inches (33 microns) for the same test . There are many ways to balance a mechanism either from a single test or combination of tests. One can not draw general conclusions about the merit of one versus others. Because offsets are independent of loads, one can establish expressions from test combinations that eliminate applied load magnitude, frequency and time.

10. CONCLUSIONS

Response coupling between degrees of freedom in flexible structural systems is formulated in the general case to aid the design of precision mechanisms. Points where loads and motions decouple provide functional advantages. Conditions for their existence and location are derived to help the design of better mechanism pointing, tracking and jitter mitigation. Because not all structures possess such characteristic points, frame-invariant measures of coupling led to the identification of stationary points at which coupling is minimum. Practical examples are used to demonstrate how to find such points and take advantage of their properties to eliminate jitter and improve performcance.

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